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# Assessing the capacity of two-flux models to predict the spectral properties of layered materials

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## Abstract

*A classical way of coloring a surface in order to create a still image is the application of a colored coating. The more recent digital printing systems enable depositing thick coatings or successive ink layers. The color rendering of the surface depends on the optical properties of the coated materials (optical index, spectral scattering and absorption coefficients) and their thickness. In order to predict its spectral reflectance as a function of these parameters, the so-called two-flux approach is to be tested in first since the model is simple and relies on analytical equations. It has a good chance to provide accurate predictions for coatings made of solid layers of strongly scattering or nonscattering media, or even complex stratified coatings obtained by stacking nonsymmetrical components such as printed films. The generalized Kubelka-Munk model summarized in this paper enables treating all these configurations with a unified mathematical formalism. But it has limitations and may provide poor color predictions for certain types of layered materials. We therefore propose a simple method based on parameters of the model to check the precision of the two-flux model for a given type of coating.*

## 1. Introduction

Predicting the color or the spectral reflectance of prints has been a subject of investigation for several decades and color management for these types of surfaces can now be done with good accuracy [1]. The challenge today is to achieve comparable prediction accuracy in the case of colored objects made of several layers or stacked component, such as surfaces printed in relief with 2.5D printer or with stacks of glass plates, which are becoming a favorite material for sculptors (see for example [2]). In order to predict the color variation of the object's color as a function of the number of material slices, one needs an accurate and efficient model taking into account, rigorously but in a simple way, the optical properties of the materials. Flux propagation models are a promising to achieve this target.

The simplest flux propagation model, to be tested in first when considering given layered material, is the two-flux model, either in its classical version proposed by Kubelka [3, 4] in the case of strongly scattering layers in optical contact with each other, or in its discrete version introduced in previous works in the case of stacks of nonscattering materials [5-8]. The continuous Kubelka-Munk formalism and the discrete formalism have been unified into a generalized two-flux model relying on a transfer matrix model that we will recall here.

The main objective of this paper is to synthetically present the most general formulas related to the generalized two-flux model and its four versions: the *continuous-symmetrical* version for uniform scattering layers yielding the classical Kubelka-Munk formulas; the *discrete-symmetrical* version for piles of identical layers, films or plates having same reflectance on their two faces; the *discrete-nonsymmetrical* version for piles of identical layers,

films or plate having different reflectances on their two faces; and the *continuous-nonsymmetrical* version for stacks of layers having different absorption and scattering coefficients according to the forward and backward flux directions. For the first time, we propose to gather the formulas associated with these four versions. A second objective of this study is to propose a simple and efficient method to check the validity of the two-flux model for a given type of layered material, thanks to the invariance of some parameters of the model according to the number of stacked components.

## 2. Generalized two-flux model

The generalized two-flux model addresses stacks of identical planar optical components such as layers of nonscattering media, layers of strongly scattering media and their eventual interfaces with air. The stacking may also be composed of more complex optical components such as colored films, alternations of ink layers, glass plates... The interesting point is that the macroscopic measurement of the reflectance and transmittance of one component suffices to predict the reflectance and transmittance of stacks of any number of similar components.

The generalized two-flux equations can be easily derived thanks to a matrix formalism, based on flux transfer matrices, presented in detail in [7]. We propose to remind the main lines of this matrix model before presenting the general reflectance and transmittance equations, and various special cases, among which the Kubelka-Munk model for uniform layers of scattering material.

### 2.1 Transfer matrix model

Let us consider a planar optical component for example a sublayer of material with unit thickness, or a film with interfaces with air... When illuminated from the front side, its reflectance is  $R$  and its transmittance is  $T$ . When illuminated on the back side, it can have a different reflectance  $R'$  and a different transmittance  $T'$ . All reflectances and transmittances are spectral quantities, even though their dependence upon wavelength is not explicitly mentioned in the following equations.

The incoming and outgoing fluxes at both sides of the component, defined in Figure 1, are related by a matrix  $\mathbf{M}$

$$\begin{pmatrix} I_0 \\ J_0 \end{pmatrix} = \mathbf{M} \begin{pmatrix} I_1 \\ J_1 \end{pmatrix}, \quad (1)$$

called the *transfer matrix* of the component, defined in terms of its reflectances and transmittances as:

$$\mathbf{M} = \frac{1}{T} \begin{pmatrix} 1 & -R' \\ R & TT' - RR' \end{pmatrix}, \quad (2)$$

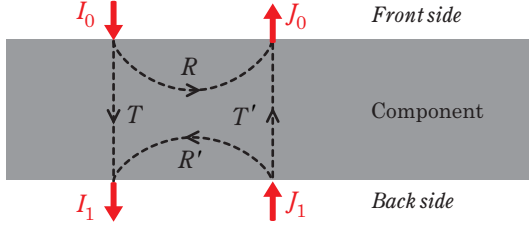


Figure 1: Flux transfers between the front and back sides of an optical component. The incoming and outgoing fluxes are represented by red arrows, and the transfers by black dashed arrows.

The diagonalization of  $\mathbf{M}$  yields:

$$\mathbf{M} = \frac{\rho}{2\beta} \begin{pmatrix} 1/\rho & 1/\rho \\ \alpha + \beta & \alpha - \beta \end{pmatrix} \cdot \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} \cdot \begin{pmatrix} -\alpha - \beta & 1/\rho \\ \alpha + \beta & 1/\rho \end{pmatrix}$$

where

$$\begin{aligned} \alpha &= \frac{1 + RR' - TT'}{2\sqrt{RR'}}, \\ \beta &= \sqrt{\alpha^2 - 1}, \\ \rho &= \sqrt{R/R'}, \\ \tau &= \sqrt{T/T'}, \\ \mu &= \frac{1}{T} [1 - (\alpha + \beta)\sqrt{RR'}], \\ \nu &= \frac{1}{T} [1 - (\alpha - \beta)\sqrt{RR'}]. \end{aligned} \quad (3)$$

Parameter  $\alpha$  generalizes the parameter denoted  $a$  in the classical Kubelka-Munk model (see Section 3.1), whose inverse is sometimes called *albedo* of the scattering medium [9-10]. Parameters  $\mu$  and  $\nu$  are the eigenvalues of matrix  $\mathbf{M}$ . They are related by the remarkable equalities:

$$\mu\nu = T'/T = 1/\tau^2.$$

When  $k$  components are stacked with each other, the stack has a reflectance  $R_k$  and a transmittance  $T_k$  when illuminated on the front side, and a reflectance  $R'_k$  and a transmittance  $T'_k$  when illuminated on the back side. The transfer matrix  $\mathbf{M}_k$  of the stack, defined as

$$\mathbf{M}_k = \frac{1}{T_k} \begin{pmatrix} 1 & -R'_k \\ R_k & T_k T'_k - R_k R'_k \end{pmatrix},$$

is simply the matrix  $\mathbf{M}$  raised to the power  $k$ . Matrix  $\mathbf{M}_k$  is therefore given by:

$$\mathbf{M}_k = \mathbf{M}^k = \frac{\rho}{2\beta} \begin{pmatrix} 1/\rho & 1/\rho \\ \alpha + \beta & \alpha - \beta \end{pmatrix} \cdot \begin{pmatrix} \mu^k & 0 \\ 0 & \nu^k \end{pmatrix} \cdot \begin{pmatrix} -\alpha - \beta & 1/\rho \\ \alpha + \beta & 1/\rho \end{pmatrix} \quad (4)$$

and the eigenvalues of  $\mathbf{M}_k$  are, for every  $k$ ,

$$\mu_k = \mu^k \quad \text{and} \quad \nu_k = \nu^k. \quad (5)$$

The reflectances and transmittances of the stack are derived from matrix  $\mathbf{M}_k = \{m_{ij}\}$  by the following formulas:

$$\begin{aligned} R_k &= m_{21} / m_{11}, \\ T_k &= 1 / m_{11}, \\ R'_k &= -m_{12} / m_{11}, \\ T'_k &= \det \mathbf{M}_k / m_{11} = m_{22} - m_{21} m_{12} / m_{11}. \end{aligned} \quad (6)$$

## 2.2 Main formulas of the generalized two-flux model

After Kubelka's early work on the stacking of diffusing layers [4], it is known that the reflectances and transmittances of a stacking of two similar components are given by:

$$\begin{aligned} R_2 &= R + \frac{TT'R}{1 - R'R}, \\ T_2 &= \frac{T^2}{1 - R'R}, \\ R'_2 &= R' + \frac{TT'R'}{1 - R'R}, \\ T'_2 &= \frac{T'^2}{1 - R'R}. \end{aligned} \quad (7)$$

They are the special case for  $k = 2$  of the following equations valid for any number  $k$  of similar components:

$$\begin{aligned} R_k &= \sqrt{\frac{R}{R'}} \frac{\nu^k - \mu^k}{\nu^k (\alpha + \beta) - \mu^k (\alpha - \beta)}, \\ T_k &= \frac{2\beta}{\nu^k (\alpha + \beta) - \mu^k (\alpha - \beta)}, \\ R'_k &= \frac{R'}{R} R_k, \\ T'_k &= \left(\frac{T'}{T}\right)^k T_k, \end{aligned} \quad (8)$$

where parameters  $\alpha$ ,  $\beta$ ,  $\rho$ ,  $\tau$ ,  $\mu$  and  $\nu$  are given in (3). Equations (8) are the results of equations (6) applied on the matrix  $\mathbf{M}_k$  given in (4). By rearranging equations (8), we also obtain the following original equations, whose similarity with the classical Kubelka-Munk reflectance and transmittance formulas can be noticed:

$$\begin{aligned} R_k &= \sqrt{\frac{R}{R'}} \cdot \frac{\sinh[k \log(\nu\tau)]}{a \sinh[k \log(\nu\tau)] + \beta \cosh[k \log(\nu\tau)]}, \\ T_k &= \frac{\beta e^{k \log \tau}}{a \sinh[k \log(\nu\tau)] + \beta \cosh[k \log(\nu\tau)]}, \\ R'_k &= (R'/R) R_k, \\ T'_k &= (T'/T)^k T_k. \end{aligned} \quad (9)$$

Equations (8) and (9) are valid for any positive integer number  $k$ , provided the two-flux theory is adapted to the considered layered material which is generally the case of the media are strongly scattering or nonscattering, and nonfluorescing. We can verify that for  $k = 2$ , they yield equations (7). By extension, we can consider any *real* number  $k$ , which makes sense for example in the case of homogeneous layers of paints: if one component represents a layer of thickness 1 mm, we have  $k = 2.34$  for layer of thickness 2.34 mm.

The parameters defined in equation (3) only depend on the reflectances and the transmittances of the elementary component. One remarkable result of the generalized two-flux theory is that  $\alpha$  and  $\rho$  remain constant if the reflectances and transmittances of one component,  $R$ ,  $R'$ ,  $T$  and  $T'$  are respectively replaced with the ones of  $k$  stacked components,  $R_k$ ,  $R'_k$ ,  $T_k$ ,  $T'_k$ , for any  $k$  (it is not the case for  $\tau$ ). We advocate that this invariance property is a way of assessing the validity of the generalized two-flux theory for a given layered material, as we will explain in Section 4.

When the number of stacked components tends to infinity, the transmittance trends to zero and the reflectance tends to a limit expressed as:

$$R_\infty = \sqrt{\frac{R}{R'}} \frac{1}{\alpha + \beta} = \sqrt{\frac{R}{R'}} (\alpha - \beta) \quad (10)$$

### 3. Special cases

The generalized two-flux model can be developed under various forms according to the considered type of layered material. One special case is a uniform layer of strongly scattering medium, illuminated by Lambertian fluxes. The model becomes equivalent to the Kubelka-Munk model, which forms a continuous version of the generalized two-flux model. A second special case is a stack of similar nonscattering components illuminated by collimated light, for example a stack of colored films; the reflectance and the transmittance of each film includes the internal transmittance of the film bulk and the flux transfers at the air-film interfaces. The number of films being an integer number, the model form a discrete version of the generalized two-flux model.

Table 1 gives a synthetic view of the formulas attached to the continuous and discrete version of the generalized two-flux model, in the case of either symmetric or nonsymmetric layers.

#### 3.1 Continuous model for homogenous scattering layer

Let us consider a homogenous layer of strongly scattering medium, of thickness  $h_0$ , illuminated by Lambertian fluxes. This layer can be considered as the elementary component of the generalized two flux model. Its reflectances  $R$  and  $R'$  are generally equal, and its transmittances  $T$  and  $T'$  are also equal: the layer is said to be symmetric. Then, thanks to the equations given in Section 2.2, we can compute by using equations (8), or equivalently by equations (9), the reflectance and the transmittance of a layer of any thickness  $h$  made with this material, by considering it as a stack of  $k$  elementary layers of thickness  $h_0$ , with  $k = h / h_0$ . In this case,  $k$  may be any positive, real number.

We can show that equations (8) or (9) are mathematically equivalent to the classical Kubelka-Munk reflectance and transmittance formulas. For that purpose, we consider a layer with thickness  $h$  for which we want to compute the reflectance  $R_h$  and transmittance  $T_h$ . We select as elementary component a sublayer with thickness  $h/k$ . The whole layer is therefore a stack of  $k$

identical sublayers. If  $k$  is large enough, the sublayer's thickness can be assumed infinitesimal. Therefore, according to the Kubelka-Munk model, its reflectance is  $Sh/k$ , where  $S$  is the scattering coefficient of the material, and its transmittance is  $1 - (K + S)h/k$ , where  $K$  is the absorption coefficient of the material. The general formulas (3) and (9) therefore apply with

$$R = R' = S \frac{h}{k}$$

$$T = T' = 1 - (K + S) \frac{h}{k}$$

Obviously, we have  $\rho = \tau = 1$ .  
As  $k$  tends to infinity, we have:

$$\lim_{k \rightarrow \infty} \alpha = \frac{K + S}{S} = a,$$

$$\lim_{k \rightarrow \infty} \beta = \sqrt{a^2 - 1} = b.$$

Moreover, knowing the following classical result for the exponent function [11]:

$$e^x = \lim_{k \rightarrow \infty} \left(1 + \frac{x}{k}\right)^k,$$

we also have:

$$\lim_{k \rightarrow \infty} k \log v = \lim_{k \rightarrow \infty} \log \frac{\left[1 - (\alpha - \beta) \frac{Sh}{k}\right]^k}{\left[1 - (K + S) \frac{h}{k}\right]^k} = \log \left[ \frac{e^{(b-a)Sh}}{e^{-aSh}} \right] = bSh.$$

By introducing the limits obtained above into equations (9), we retrieve the classical Kubelka-Munk formulas:

$$R_h = \lim_{k \rightarrow \infty} R_k = \frac{\sinh(bSh)}{a \sinh(bSh) + \beta \cosh(bSh)}, \quad (11)$$

$$T_h = \lim_{k \rightarrow \infty} T_k = \frac{b}{a \sinh(bSh) + \beta \cosh(bSh)}.$$

When the layer's thickness tends to infinity, the general reflectance expression given by (10) simplifies as

$$R_\infty = a - b \quad (12)$$

which is a famous result of the Kubelka-Munk theory.

#### 3.2 Continuous model for nonsymmetrical scattering layers

Even though it is physically difficult to conceive, the mathematical model enables imagining a scattering medium in which an infinitesimal layer would have different reflectances and transmittances according to the illuminated side. This would result in different absorption and scattering coefficients for the forward flux (respectively denoted  $K$  and  $S$ ) and the backward flux ( $K'$  and  $S'$ ). As shown in appendix of [7], the Kubelka-Munk formulas easily extend to this configuration.

**Table 1. Main formulas of the generalized two-flux model .**

Model version	Discrete-symmetrical		Discrete-nonsymmetrical		Continuous-symmetrical (Kubelka-Munk model)		Continuous-nonsymmetrical
<i>Single component</i>	Symmetrical component		Non-symmetrical component		Sublayer (thickness $h/k$ )		Nonsymmetrical sublayer (thickness $h/k$ )
Coefficients					Absorption: $K$ Scattering: $S$		Absorption: $K, K'$ Scattering: $S, S'$
Reflectance and transmittance of one component	$R$ $T$		$R, R'$ $T, T' \quad (\tau = \sqrt{T/T'})$		$R = S \frac{h}{N}, \quad T = 1 - (K + S) \frac{h}{N}$ $R' = S' \frac{h}{N}, \quad T' = 1 - (K' + S') \frac{h}{N}$		
Transfer matrix of one component	$\frac{1}{T} \begin{pmatrix} 1 & -R \\ R & T^2 - R^2 \end{pmatrix}$		$\frac{1}{T} \begin{pmatrix} 1 & -R' \\ R' & TT' - RR' \end{pmatrix}$		$\frac{1}{T} \begin{pmatrix} 1 & -R \\ R & T^2 - R^2 \end{pmatrix}$		$\frac{1}{T} \begin{pmatrix} 1 & -R' \\ R' & TT' - RR' \end{pmatrix}$
Eigenvalues of the transfer matrix	$\mu = \frac{1}{T} [1 - R(\alpha + \beta)]$ $\nu = \frac{1}{T} [1 - R(\alpha - \beta)]$		$\mu = \frac{1}{T} [1 - \sqrt{RR'}(\alpha + \beta)]$ $\nu = \frac{1}{T} [1 - \sqrt{RR'}(\alpha - \beta)]$		$\mu = \frac{1 - S(\alpha + \beta)h/k}{1 - (K + S)h/k}$ $\nu = \frac{1 - S(\alpha - \beta)h/k}{1 - (K + S)h/k}$		$\mu = \frac{1 - \sqrt{SS'}(\alpha + \beta)h/k}{1 - (K + S')h/k}$ $\nu = \frac{1 - \sqrt{SS'}(\alpha - \beta)h/k}{1 - (K + S')h/k}$
$\alpha$	$\alpha = \frac{1 + R^2 - T^2}{2R}$		$\alpha = \frac{1 + RR' - TT'}{2\sqrt{RR'}}$		$a = \lim_{N \rightarrow \infty} \frac{K + S}{S}$		$a = \lim_{N \rightarrow \infty} \frac{K + K' + S + S'}{2\sqrt{SS'}}$
$\beta$	$\beta = \sqrt{\alpha^2 - 1}$		$\beta = \sqrt{\alpha^2 - 1}$		$b = \sqrt{a^2 - 1}$		$b = \sqrt{a^2 - 1}$
<b>Stacked components</b>	Stack of $k$ components		Stack of $k$ components		Layer of thickness $h$ ( $k$ sublayers, with $k \rightarrow \infty$ )		Layer of thickness $h$ ( $k$ sublayers, with $k \rightarrow \infty$ )
Reflectance and transmittance of the stack	$R_k = \frac{\sinh[k \log \nu]}{\alpha \sinh[k \log \nu] + \beta \cosh[k \log \nu]}$ $T_k = \frac{\beta}{\alpha \sinh[k \log \nu] + \beta \cosh[k \log \nu]}$		$R_k = \frac{\sinh[k \log(\nu\tau)] \sqrt{R'/R}}{\alpha \sinh[k \log(\nu\tau)] + \beta \cosh[k \log(\nu\tau)]}$ $T_k = \frac{\beta e^{k \log \tau}}{\alpha \sinh[k \log(\nu\tau)] + \beta \cosh[k \log(\nu\tau)]}$		$R_h = \frac{\sinh(bSh)}{\alpha \sinh(bSh) + b \cosh(bSh)}$ $T_h = \frac{b}{\alpha \sinh(bSh) + b \cosh(bSh)}$		$R_h = \frac{\sinh(b\sqrt{SS'}h) \sqrt{S/S'}}{\alpha \sinh(b\sqrt{SS'}h) + b \cosh(b\sqrt{SS'}h)}$ $T_h = \frac{b e^{-(K+S-K'-S')h}}{\alpha \sinh(b\sqrt{SS'}h) + b \cosh(b\sqrt{SS'}h)}$
			$R'_k = (R'/R)R_k$ $T'_k = (T'/T)^k T_k$				$R'_h = (S'/S)R_h$ $T'_h = e^{-2(K'+S'-\alpha\sqrt{SS'})h} T_h$
Reflectance of a semi-infinite stack	$R_\infty = \alpha - \beta = \frac{1}{\alpha + \beta}$		$R_\infty = \sqrt{R'/R}(\alpha - \beta) = \frac{\sqrt{R'/R}}{\alpha + \beta}$		$R_\infty = a - b = \frac{1}{a + b}$		$R_\infty = \sqrt{S/S'}(a - b) = \frac{\sqrt{S/S'}}{a + b}$

The parameters defined in (3) become:

$$\begin{aligned}\alpha &= \frac{K + K' + S + S'}{2\sqrt{SS'}}, \\ \beta &= \sqrt{\alpha^2 - 1}, \\ \rho &= \sqrt{S/S'}, \\ \tau &= \sqrt{T/T'}, \\ \mu &= e^{[K+S-\sqrt{SS'}(\alpha+\beta)]h}, \\ \nu &= e^{[K+S-\sqrt{SS'}(\alpha-\beta)]h}.\end{aligned}$$

and the reflectance and transmittance expressions for a layer of thickness  $h$  become:

$$\begin{aligned}R_h &= \frac{\sinh[\beta\sqrt{SS'}h]\sqrt{S/S'}}{\alpha\sinh[\beta\sqrt{SS'}h] + \beta\cosh[\beta\sqrt{SS'}h]}, \\ T_h &= \frac{\beta e^{-(K+S-\alpha\sqrt{SS'})h}}{\alpha\sinh[\beta\sqrt{SS'}h] + \beta\cosh[\beta\sqrt{SS'}h]}, \\ R'_h &= \frac{S'}{S}R_h, \\ T'_h &= e^{-2(K'+S'-\alpha\sqrt{SS'})h}T_h.\end{aligned}\quad (13)$$

### 3.3 Generalized Saunderson correction

In sections 3.1 and 3.2, we considered a stacking of planar diffusing components. The model does not apply as it is if the first and last components have an interface with air while the other components do not: due to the optical effect of the interfaces, the bordering components could not be considered as similar to the other ones. This is the case for example when stacking several layers of a same paint. The optical effect of the bordering interfaces is to be considered separately, in a similar manner as the correction by Saunderson of the Kubelka-Munk model [12].

The reflectances and transmittances of the interfaces at the air-side and the medium-side are computed by using the Fresnel formulas according to the angular distribution of the incoming light and the geometry of the detectors. Table 1 in [7] gives these reflectances and transmittances for different geometries. Let us consider here that the fluxes coming from air are Lambertian, and that the whole outgoing light in air is collected by an integrating sphere (so called diffuse-diffuse geometry, denoted d:d according to the recommendations by the CIE). By denoting as  $R_{01}(\theta)$  the Fresnel reflectance of the interface for a light ray coming from air at angle  $\theta$ , and as  $R_{10}(\theta')$  the one for a light ray coming from the medium at angle  $\theta'$ , the air-side reflectance of the interface,  $r_s$ , the air-to-medium transmittance  $T_{in}$ , the medium-to-air transmittance  $T_{ex}$ , and the medium-side reflectance  $r_i$ , are respectively given by:

$$\begin{aligned}r_s &= \int_{\theta=0}^{\pi/2} R_{01}(\theta) \sin 2\theta d\theta \\ T_{in} &= \int_{\theta=0}^{\pi/2} [1 - R_{01}(\theta)] \sin 2\theta d\theta \\ T_{ex} &= \int_{\theta'=0}^{\pi/2} [1 - R_{10}(\theta')] \sin 2\theta' d\theta' \\ r_i &= \int_{\theta'=0}^{\pi/2} R_{10}(\theta') \sin 2\theta' d\theta'\end{aligned}$$

With the interfaces, the stack of layers, whose intrinsic reflectances and transmittances are given by (13) in the most general case, has now the reflectances and transmittances (denoted with a capping  $\sim$  symbol):

$$\begin{aligned}\tilde{R}_h &= r_s + T_{in}T_{ex} \frac{R_h - r_i(R_h R'_h - T_h T'_h)}{(1 - r_i R_h)(1 - r_i R'_h) - r_i^2 T_h T'_h} \\ \tilde{T}_h &= \frac{T_{in}T_{ex}T_h}{(1 - r_i R_h)(1 - r_i R'_h) - r_i^2 T_h T'_h}, \\ \tilde{R}'_h &= r_s + T_{in}T_{ex} \frac{R'_h - r_i(R_h R'_h - T_h T'_h)}{(1 - r_i R_h)(1 - r_i R'_h) - r_i^2 T_h T'_h}, \\ \tilde{T}'_h &= \frac{T_{in}T_{ex}T'_h}{(1 - r_i R_h)(1 - r_i R'_h) - r_i^2 T_h T'_h}.\end{aligned}\quad (14)$$

These reflectance and transmittance expressed by (14) are the quantities which can be measured. From them, by using the following equations, we can deduce the intrinsic reflectances and transmittances of the stack of layers:

$$\begin{aligned}R_h &= \frac{1}{\Delta} \{ (\tilde{R}_h - r_s) [T_{in}T_{ex} - r_i(\tilde{R}'_h - r_s)] + r_i \tilde{T}_h \tilde{T}'_h \}, \\ R'_h &= \frac{1}{\Delta} \{ (\tilde{R}'_h - r_s) [T_{in}T_{ex} - r_i(\tilde{R}_h - r_s)] + r_i \tilde{T}_h \tilde{T}'_h \}, \\ T_h &= \frac{T_{in}T_{ex}\tilde{T}_h}{\Delta}, \\ T'_h &= \frac{T_{in}T_{ex}\tilde{T}'_h}{\Delta}.\end{aligned}\quad (15)$$

with

$$\Delta = [T_{in}T_{ex} - r_i(\tilde{R}_h - r_s)][T_{in}T_{ex} - r_i(\tilde{R}'_h - r_s)] - r_i^2 \tilde{T}_h \tilde{T}'_h.$$

### 3.4 Discrete model for stacks of symmetrical nonscattering components

The generalized two-flux model also applies with stacks of nonscattering components illuminated by collimated light. These components may be clear or colored glass plates or polymer films.

The reflectance and transmittance of one plate considered alone result from a multiple reflection process of light between the two air-medium interfaces. When stacking the plates, a slice of air should remain between them in order to be sure that each plate has similar reflectance and transmittance as a single one. This is the case when the plates are simply deposited on top of each other, without optical contact. If a liquid is used to make the optical contact between the plates, the present model does not apply because the relative refractive index of the interfaces, thereby the multiple reflection process within the plates is modified.

Experiments with stacks of nonscattering colored acetate films, illuminated by collimated light according to their normal direction, have already been presented in [13] and [8]. Other experiments based on transparency films printed in inkjet, with different reflectances on the inked and noninked faces due to a bronzing effect [14], have also been presented in [15]. In these studies, other mathematical methods than the present one were used to derive the reflectance and transmittance equations of stacks of films, which were however mathematically equivalent to equations (8).

We can mention here the reflectance and transmittance expressions corresponding to a stack of  $k$  symmetrical plates, for which we have:  $R = R'$  and  $T = T'$ . Equations (8) can be written as:

$$R_k = R'_k = \frac{1}{\alpha - \beta \left( 1 - \frac{2}{1 - \left[ \frac{1 - (\alpha + \beta)R}{1 - (\alpha - \beta)R} \right]^k} \right)},$$

and

$$T_k = T'_k = \frac{2\beta T^k}{(\alpha + \beta)[1 - (\alpha - \beta)R]^k - (\alpha - \beta)[1 - (\alpha + \beta)R]^k},$$

with

$$\alpha = \frac{1 + R^2 - T^2}{2R} \quad \text{and} \quad \beta = \sqrt{\alpha^2 - 1}.$$

In the case of nonscattering components, the two-flux model applies only with collimated light (for any incidence angle in air), due to the principle of directionality stated in [16]. If the incident light is Lambertian, the reflectance of the stack of plates,  $r_k$ , is obtained by integrating its angular reflectance  $R_k(\theta)$ , expressed as a function of the incident angle or measured for all angles from 0 to  $\pi/2$ :

$$r_k = \int_{\theta=0}^{\pi/2} R_k(\theta) \sin 2\theta d\theta$$

The reflectance at the back side, and the front-side and back-side transmittances are given by similar integrals, where  $R_k(\theta)$  is respectively replaced with  $R'_k(\theta)$ ,  $T_k(\theta)$  and  $T'_k(\theta)$ .

#### 4. Using the invariant parameters to check the validity of the two-flux model

The reflectance and the transmittance of a stack of similar components generally vary as a function of the number  $k$  of stacked components, whereas parameter  $\alpha$  of the model is independent of  $k$ . Indeed, we can show that for every  $k$  value, we have:

$$\alpha_k = \frac{1 + R_k R'_k - T_k T'_k}{2\sqrt{R_k R'_k}} \equiv \frac{1 + RR' - TT'}{2\sqrt{RR'}} = \alpha \quad (16)$$

where  $R$ ,  $R'$ ,  $T$ , and  $T'$  are the reflectances and transmittances attached to one components, and  $R_k$ ,  $R'_k$ ,  $T_k$ , and  $T'_k$  the ones attached to  $k$  components. According to (5), we also have, for every  $k$  value:

$$\mu_k^{1/k} = \frac{1}{T_k} \left[ 1 - (\alpha_k + \beta_k) \sqrt{R_k R'_k} \right] \equiv \frac{1}{T} \left[ 1 - (\alpha + \beta) \sqrt{RR'} \right] = \mu. \quad (17)$$

The aim of this section is to see how these invariant properties can help to estimate whether the two-flux model is able to provide good reflectance and transmittance predictions for a given type of

layered material. For that purpose, we selected sheets of various materials, stacked them incrementally, and measured their reflectance and transmittance.

#### Nonscattering materials

The following nonscattering materials were tested: sheets of cyan acetate (1 to 5 sheets, similar as the ones used for [13]), and transparency films printed by inkjet with a green or a magenta color (1 to 16 sheets, similar as the ones used for [15]). Their reflectances and transmittances, displayed in Figure 2-a, respectively Figure 2-c, were measured by using the X-rite Colori7 spectro-photometer based on the diffuse:0° geometry. However, since these materials are nonscattering, all incident radiances except the one normal to the samples are ignored; the effective geometry is therefore the 0°:0° geometry [17].

Figure 2 gathers various graphics issued from our experiments with these three types of materials. Each column of graphics corresponds to one material, and contains the following graphs.

In row (a) are plotted the spectral front-side reflectances of the materials (the back-side reflectances are not displayed in this figure). The black solid line corresponds to the measured spectral reflectance of one sheet, and the red solid lines to the ones of  $k \geq 2$  sheets. The reflectance progressively increases as  $k$  increases. The black dashed lines correspond to the spectral reflectances of  $k \geq 2$  sheets predicted by equations (9) from the measured reflectances and transmittances of one sheet. The graphs in row (b) give for each stack the deviation between the measured and predicted spectral reflectances, assessed by the visual metric CIELAB  $\Delta E_{94}$  (obtained by converting both spectra into CIE-1931 XYZ tristimulus values then into CIELAB color coordinates, by using the D65 illuminant as a white reference). The model is fairly accurate with these materials since the  $\Delta E_{94}$  value does not exceed 0.5 unit.

In row (c) are shown the measured and predicted spectral transmittances of the stacks. Similar color code as for reflectance is used to indicate measured and predicted spectra. The transmittance progressively decreases as the number of stacked sheets increases. For each stack, the  $\Delta E_{94}$  values assessing the deviation between the measured and predicted transmittance spectra are displayed in row (d). The predictions for the blue acetate sheets and the green printed films are satisfying, even though they become poorer when the number of stacked films is too high, mainly because the stack is opaque in some wavelength domains, thus more sensible to measurement noise. The problem of opacity is more striking with the magenta printed films, more absorbing than the two other ones; the spectral transmittance predictions are clearly less accurate than for the other films.

In row (e), the spectral  $\alpha_k$  parameter is computed according to (16) for all  $k \geq 1$ . As expected, we observe that the spectral  $\alpha_k$  parameter is almost independent of  $k$ : the different plotted curves are very close to each other. The rms deviation between the spectral curves of  $\alpha_k$  ( $k \neq 1$ ) and  $\alpha$  ( $k = 1$ ) is shown in the graphs of row (f). It is lower than 2% for all tested samples.

In row (g), the spectral parameter  $\mu_k^{1/k}$  is computed according to formula (17) for all  $k \geq 1$ . As  $\alpha_k$ , it is almost independent of  $k$  except small deviations around 550 nm for the magenta printed films, most probably due to the fact that the transmittance  $T_k(\lambda)$  trends to zero in this spectral domain as  $k$  increases. The rms deviation between the spectral curves of  $\mu_k^{1/k}$  ( $k \neq 1$ ) and  $\mu$  ( $k = 1$ ), shown in the graphs of row (h), is again below 2% except for the magenta films where it is much higher for the reason mentioned above.

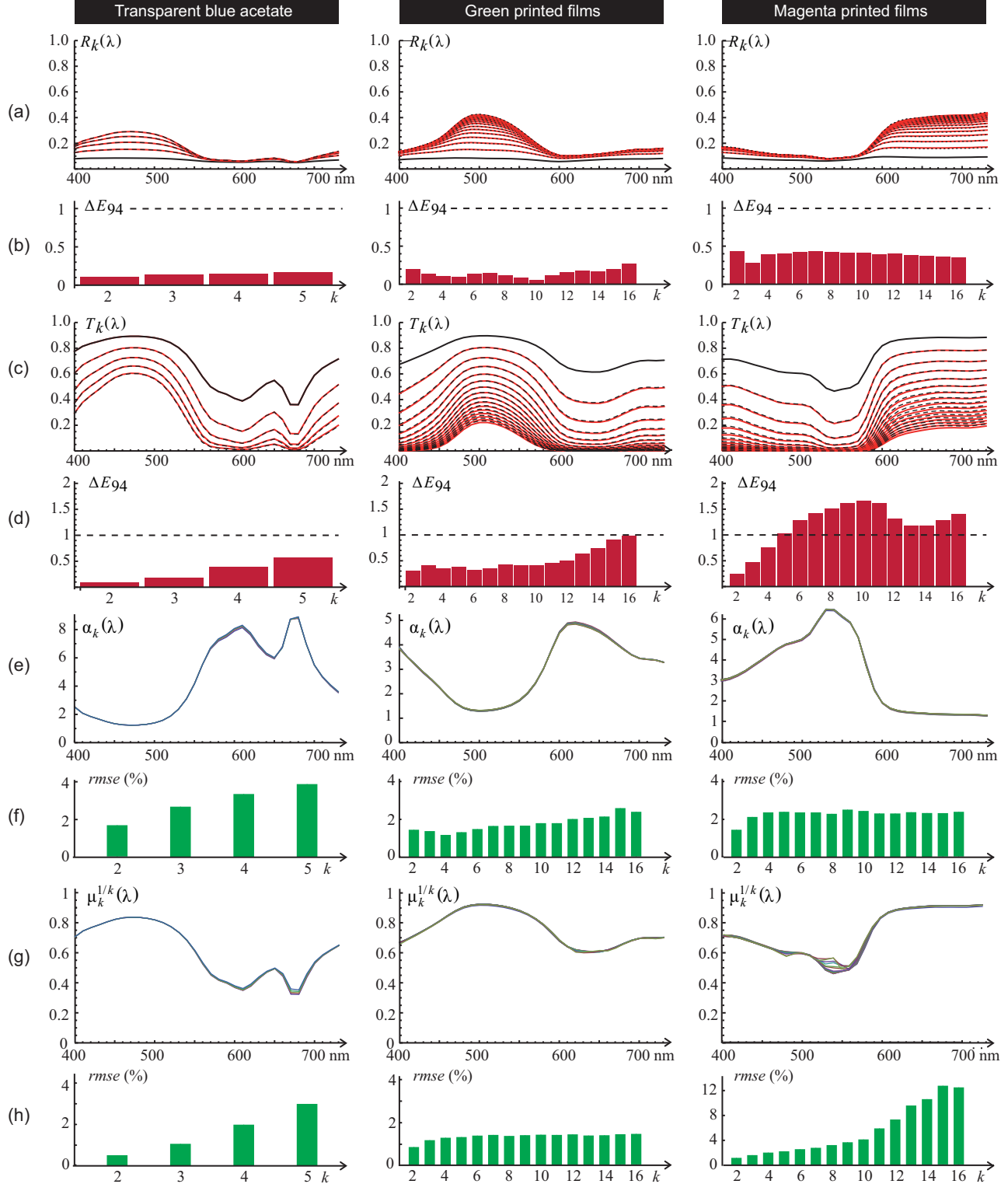


Figure 2: Spectral properties of stacks of  $k$  nonscattering sheets: transparent blue acetate (left column), and transparency films printed with a green color (middle column) and with a magenta color (right column); (a) Measured (solid line) and predicted (dashed line) front-side reflectances for an incremented number of sheets; (b) CIELAB  $\Delta E_{94}$  color difference computed from the predicted and measured spectral reflectances for the different stacks of  $k$  sheets; (c) Measured (solid line) and predicted (dashed line) transmittances for an incremented number of sheets; (d) CIELAB  $\Delta E_{94}$  color difference computed from the predicted and measured spectral transmittances for the different stacks of  $k$  sheets; (e) spectral curve of the parameter  $\alpha_k$  computed for the different numbers  $k$  of sheets; (f) rms deviation between the spectral curve of the parameter  $\alpha_k$  ( $k \neq 1$ ) and the one of  $\alpha$  ( $k = 1$ ); (g) spectral curve of the parameter  $\mu_k^{1/k}$  computed for the different numbers  $k$  of sheets; (h) rms deviation between the spectral curve of the parameter  $\mu_k^{1/k}$  ( $k \neq 1$ ) and the one of  $\mu$  ( $k = 1$ ).



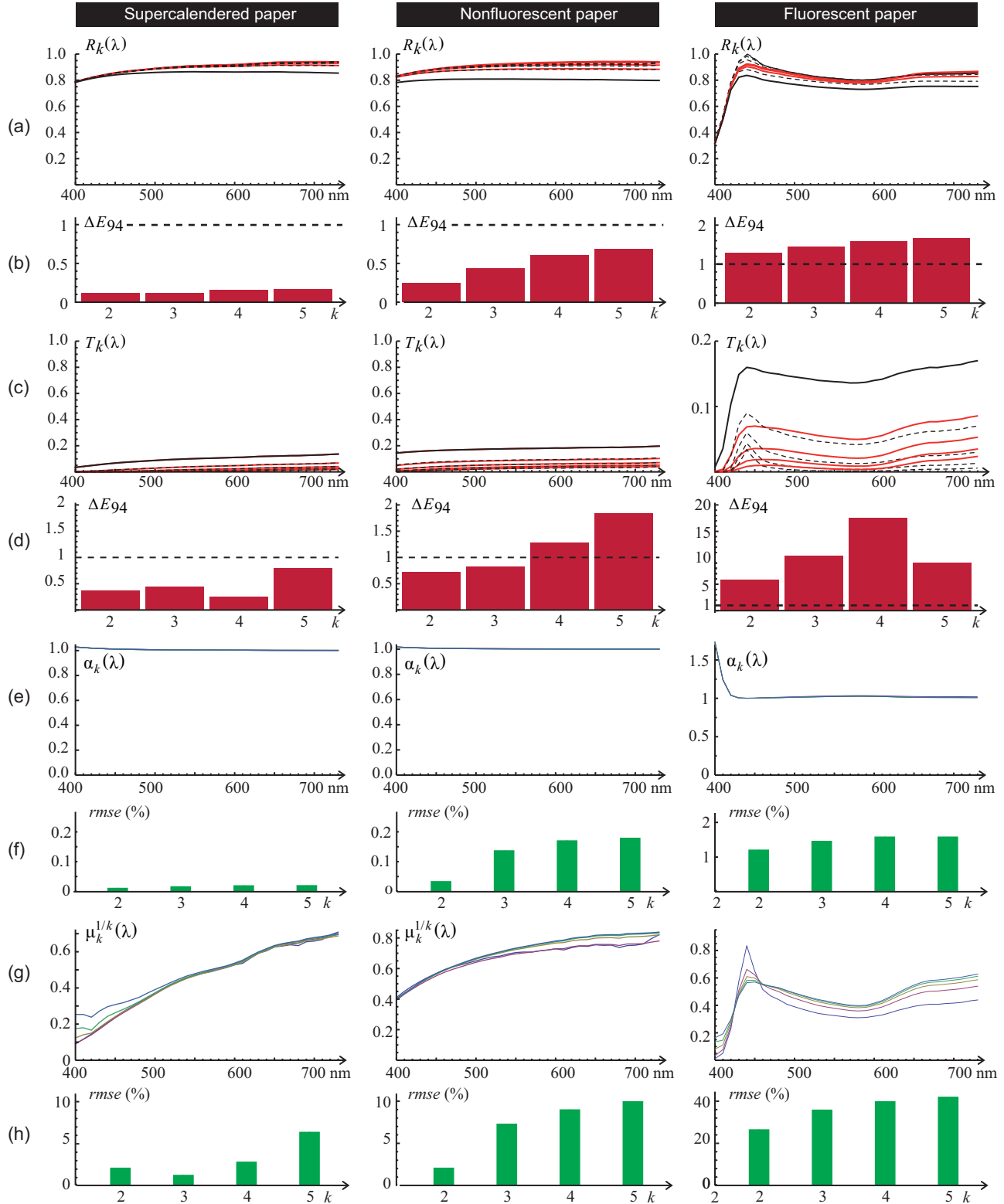


Figure 3: Spectral properties of stacks of  $k$  scattering sheets: supercalendered high quality paper (left column), nonfluorescent office paper (middle column) and usual, fluorescent office paper (right column); (a) Measured (solid line) and predicted (dashed line) front-side reflectances for an incremented number of sheets; (b) CIELAB  $\Delta E_{94}$  color difference computed from the predicted and measured spectral reflectances for the different stacks of  $k$  sheets; (c) Measured (solid line) and predicted (dashed line) transmittances for an incremented number of sheets; (d) CIELAB  $\Delta E_{94}$  color difference computed from the predicted and measured spectral transmittances for the different stacks of  $k$  sheets; (e) spectral curve of the parameter  $\alpha_k$  computed for the different numbers  $k$  of sheets; (f) rms deviation between the spectral curve of the parameter  $\alpha_k$  ( $k \neq 1$ ) and the one of  $\alpha_1$  ( $k = 1$ ); (g) spectral curve of the parameter  $\mu_k^{1/k}$  computed for the different numbers  $k$  of sheets; (h) rms deviation between the spectral curve of the parameter  $\mu_k^{1/k}$  ( $k \neq 1$ ) and the one of  $\mu_1$  ( $k = 1$ ).

## Scattering materials

Strongly scattering material were also tested: a nonfluorescent super-calendered high quality paper (1 to 5 sheets), a nonfluorescent office paper (1 to 5 sheets) which was less homogeneous than the supercalendered paper especially in transmission, and a usual office paper containing optical brighteners. The reflectances and transmittances were also measured with the Colori7 instrument; in this case the measuring geometry is the diffuse:0° geometry, almost equivalent to the diffuse:diffuse geometry since the sample are almost Lambertian. Similar graphs as for the nonscattering sheets are shown in Figure 3.

Regarding first the two types of nonfluorescent papers, we can notice that the reflectance predictions are good: the  $\Delta E_{94}$  values between the predicted and measured spectral reflectances are low, as well as the deviations between the spectral curves of  $\alpha_k$  computed for the different numbers  $k$  of sheets and the one of  $\alpha$ . Regarding the transmittance, predictions are poorer and we observe important deviations between the spectral curves for  $\mu_k^{1/k}$  ( $k \neq 1$ ) and the one for  $\mu$  ( $k=1$ ). It seems that there is a correlation between poor transmittance predictions and important variations of  $\mu_k^{1/k}$  according to  $k$ .

Regarding the usual office paper, reflectance predictions are rather poor due to the fluorescence which is not taken into account into the model. Since the first sheet in the stack absorbs the UV, the next sheets do not receive UV and therefore do not emit fluorescence. We can be surprised that, despite errors due to the fluorescence, the spectral curve of  $\alpha_k$  remains rather invariant according to  $k$ . Oppositely, the spectral curve of  $\mu_k^{1/k}$  strongly varies as a function of  $k$ , with a pick near 450 nm which is characteristic of the fluorescence. This shows that the two-flux model is clearly not adapted to fluorescing materials.

As for the nonscattering sheets, we can notice a correlation between poor reflectance predictions and non-invariance of the  $\alpha_k$  parameter according to the number  $k$  of stacked sheets, as well as a correlation between poor transmittance predictions and non-invariance of the  $\mu_k^{1/k}$  parameter. Observing the invariance of the  $\alpha_k$  and  $\mu_k^{1/k}$  parameters therefore seems to be a way to assess the capacity of the two-flux model to predict accurately the reflectance and transmittance of stacked similar components. The question is the threshold beyond which we must consider that the invariance is not satisfying. Experiments with many materials would be necessary to determine relevant, universal threshold values, but it appears through our experiments that the threshold is different for nonscattering and scattering materials. In the case of nonscattering material, a rms deviation of 2% of the  $\alpha_k$  parameter from  $\alpha$  ( $k=1$ ) does not prevent good reflectance predictions for stacks of up to ten components. However, a deviation of 2% of the  $\mu_k^{1/k}$  parameter compromises accurate stack transmittance predictions. In the case of scattering materials, a rms deviation of 1% of the  $\alpha_k$  parameter from  $\alpha$  may indicate poor reflectance predictions, and the threshold for  $\mu_k^{1/k}$  parameter seems to be around 5%.

## 5. Conclusions

This paper gathers and summarized the recent progresses of research on the two-flux theory, including especially the case of the nonscattering component in which the back-reflection of flux is due to the interfaces. One important result is the possibility to address, within a mathematical framework related to the Kubelka-Munk model, stacks of similar non-symmetrical components such as stacks of printed films or coated glass plates. Another interesting result is the fact that a parameter of the model, here

denoted  $\alpha$ , can be defined, with the same formula, from reflectances and transmittances on both sides of either one component or a stack of similar components. We can easily check from spectral measurements that this invariance property is satisfied for a given type of print. When it is not, the model will most probably provide poor reflectance prediction accuracy. Another parameter, denoted  $\mu$ , can also be defined from the spectral reflectances and transmittances of either one component or several stacked components. This parameter itself depends on the number  $k$  of components, but once raised to the power  $1/k$ , it becomes independent of  $k$ . When this invariant property is not satisfied, the model is most probably not able to predict correctly the transmittance of stacks of the considered components. From our first experiments, which should be confirmed by many other ones, we proposed empirical thresholds permitting to assess whether these parameters can be considered as invariant or not, therefore whether the two-flux model has a chance to apply for the considered material or more advance models should be used.

In the future, we would like to verify the accuracy of the generalized two-flux model with stacks of halftone ink layers, such as those that 2.5D printers can produce [18]. We can expect good accuracy of the model with solid layers of nonscattering inks and varnishes, or strongly scattering inks (e.g. white ink), or even periodical alternations of solid layers of nonscattering and strongly scattering inks, with a Saunderson correction in order to take into account the air-ink interfaces and possibly the printing support. But we might expect that the model fails in the case of halftone ink layers due to optical dot gain within each layer, especially with the diffusing ink which for a layer has transparent areas, that light crosses directionally, and diffusing areas where light is both reflected and diffusely transmitted. The extension of the model to halftone layered materials is therefore the next challenge to address. The four-flux approach [19-21], which relies on two directional fluxes and two diffuse fluxes and has been recently formulated with a similar transfer matrix formalism [22] as the one recalled in this paper, is promising since it could enable the combination of the directional and diffused flux propagation through the partly-diffusing and partly-nonscattering layers.

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